Cavalieri's delicatessen usually produced bologna in cylindrical form, so that the volume would be computed as $\pi \cdot \text{radius}^2 \cdot \text{length}$. One day, the casings were a bit weak, and the bologna came out with odd bulges. The scale was not working that day, either, so the only way to compute the price of the bologna was in terms of its volume.

Cavalieri took his best knife and sliced the bologna into n very thin slices, each of thickness Δx, and measured the radii $r_1, r_2, \ldots, r_n$ of the slices (fortunately, they were round). He then estimated the volume to be $\sum_{i=1}^{n} \pi r_i^2 \Delta x_i$, the sum of the volumes of the slices.

Cavalieri was moonlighting from his regular job as a professor at the University of Bologna. That afternoon, he went back to his desk and began the book "Geometria indivisibilium continuorum nova quandum ratione promota" ("Geometry shows the continuous indivisibility between new rations and getting promoted"), in which he stated what is now known[4] as *Cavalieri's principle*:

> *If two solids are sliced by a family of parallel planes in such a way that corresponding sections have equal areas, then the two solids have the same volume.*

The book was such a success that Cavalieri sold his delicatessen and retired to a life of occasional teaching and eternal glory.

[4] Honest!

Exercises for Section 9.1

In Exercises 1–4, use the slice method to find the volume of the indicated solid.

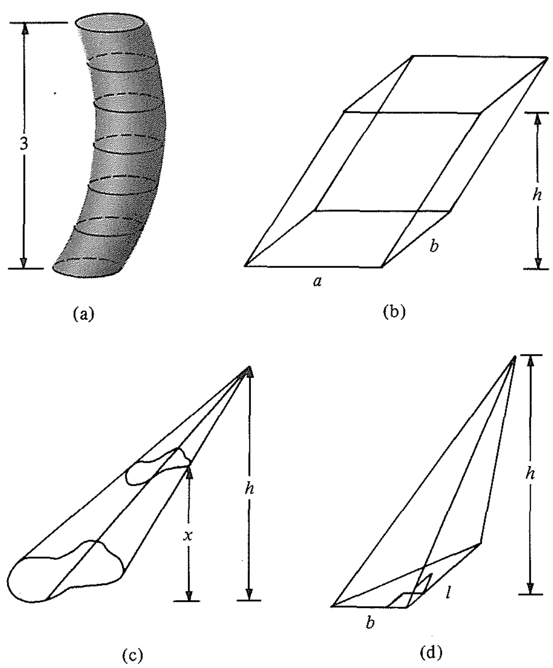

Figure 9.1.15. The solids for Exercises 1–4.

1. The solid in Fig. 9.1.15(a); each plane section is a circle of radius 1.
2. The parallelepiped in Fig. 9.1.15(b); the base is a rectangle with sides a and b.
3. The solid in Fig. 9.1.15(c); the base is a figure of area A and the figure at height x has area $A_x = [(h-x)/h]^2 A$.
4. The solid in Fig. 9.1.15(d); the base is a right triangle with sides b and l.
5. Find the volume of the tent in Fig. 9.1.16. The plane section at height x above the base is a square of side $\frac{1}{6}(6-x)^2 - \frac{1}{6}$. The height of the tent is 5 feet.

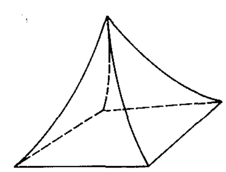

Figure 9.1.16. Find the volume of this tent.

6. What would the volume of the tent in the previous exercise be if the base and cross sections were equilateral triangles instead of squares (with the same side lengths)?
7. The base of a solid S is the disk in the xy plane with radius 1 and center $(0,0)$. Each section of S cut by a plane perpendicular to the x axis is an equilateral triangle. Find the volume of S.

8. A plastic container is to have the shape of a truncated pyramid with upper and lower bases being squares of side length 10 and 6 centimeters, respectively. How high should the container be to hold exactly one liter (= 1000 cubic centimeters)?
9. The conical solid in Fig. 9.1.6 is to be cut by horizontal planes into four pieces of equal volume. Where should the cuts be made? [*Hint:* What is the volume of the portion of the cone above the plane P_x?]
10. The tent in Exercise 5 is to be cut into two pieces of equal volume by a plane parallel to the base. Where should the cut be made?
    (a) Express your answer as the root of a fifth-degree polynomial.
    (b) Find an approximate solution using the method of bisection.
11. A wedge is cut in a tree of radius 0.5 meter by making two cuts to the tree's center, one horizontal and another at an angle of 15° to the first. Find the volume of the wedge.
12. A wedge is cut in a tree of radius 2 feet by making two cuts to the tree's center, one horizontal and another at an angle of 20° to the first. Find the volume of the wedge.
13. Find the volume of the solid in Fig. 9.1.17(a).
14. Find the volume of the solid in Fig. 9.1.17(b).

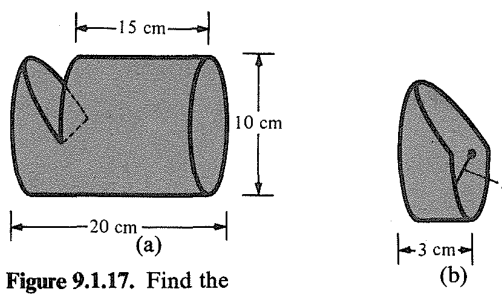

Figure 9.1.17. Find the volumes of these solids.

In Exercises 15–26, find the volume of the solid obtained by revolving each of the given regions about the x axis and sketch the region.

15. The region under the graph of $3x + 1$ on $[0, 2]$.
16. The region under the graph of $2 - (x-1)^2$ on $[0, 2]$.
17. The region under the graph of $\cos x + 1$ on $[0, 2\pi]$.
18. The region under the graph of $\cos 2x$ on $[0, \pi/4]$.
19. The region under the graph of $x(x-1)^2$ on $[1, 2]$.
20. The region under the graph of $\sqrt{4 - 4x^2}$ on $[0, 1]$.
21. The semicircular region with center $(a, 0)$ and radius r (assume that $0 < r < a, y \geqslant 0$).
22. The region between the graphs of $\sqrt{3 - x^2}$ and $5 + x$ on $[0, 1]$. (Evaluate the integral using geometry or the tables.)
23. The square region with vertices $(4, 6)$, $(5, 6)$, $(5, 7)$, and $(4, 7)$.
24. The region in Exercise 23 moved 2 units upward.
25. The region in Exercise 23 rotated by 45° around its center.
26. The triangular region with vertices $(1, 1)$, $(2, 2)$, and $(3, 1)$.

★27. A vase with axial symmetry has the cross section shown in Fig. 9.1.18 when it is cut by a plane through its axis of symmetry. Find the volume of the vase to the nearest cubic centimeter.

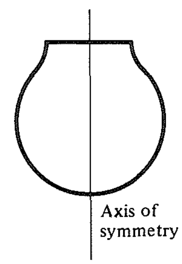

Figure 9.1.18. Cross section of a vase.

★28. A right circular cone of base radius r and height 14 is to be cut into three equal pieces by parallel planes which are parallel to the base. Where should the cuts be made?

★29. Find the formula for the volume of a doughnut with outside radius R and a hole of radius r.

★30. Use the fact that the area of a disk of radius r is $\pi r^2 = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$ to compute the area inside the ellipse $y^2/4 + x^2 = r^2$.

★31. Prove Cavalieri's principle.

★32. Using Cavalieri's principle, without integration, find a relation between the volumes of:
    (a) a hemisphere of radius 1;
    (b) a right circular cone of base radius 1 and height 1;
    (c) a right circular cylinder of base radius 1 and height 1.
    [*Hint:* Consider two of the solids side by side as a single solid. The sum of two volumes will equal the third.]